General Relativity $\boxed{\text{Week 7}}$

Last week: We saw that special relativity $\sim$ geometry of $(\mathbb{R}^{3+1}, \eta)$

Inertial observers $\Leftrightarrow$ timelike straight lines & orthonormal frame

And that Einstein decided to incorporate gravity based on two principles:

① Principle of equivalence: Free falling observers: Locally the same physical laws as inertial ones.

② Principle of general covariance

Physical laws should be "invariant" under changes of "frame"

Unifying set-up: Spacetime is (M^{3+1}, g) Lorentzian.

Free-falling observers $\rightsquigarrow$ timelike geodesics

Local frame $\rightsquigarrow$ Local coordinate system

① is automatically true: In normal coordinates around p geodesics are straight lines and $g \approx \eta$ to 2^{nd} order

②: Translates to "Physical laws should be of tensorial (aka geometric) nature"

How does matter/energy tell the spacetime how to curve?

In Newtonian gravity: Free-falling observer $t \longrightarrow (x^1(t), x^2(t), x^3(t))$

$$\begin{cases} \ddot{x}^i = -\nabla \Phi & \leftarrow \text{Grav. potential} \\ \Delta_{\mathbb{R}^3} \Phi = 4\pi \rho & \leftarrow \text{mass-density} \end{cases}$$

$\uparrow$ Poisson's eqn

In general relativity:

The 1^{st} eqn should become the geodesic eqn:

Free-falling observer $s \longrightarrow (x^0(s), x^1(s), x^2(s), x^3(s))$

$$\ddot{x}^\mu = -\Gamma^\mu_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta$$

Since $\Gamma \sim \partial g \Rightarrow \Phi \sim g$

What would be the analogue of the Poisson eqn?

In special relativity: mass/energy/momentum are <u>not</u> separately invariant under changes of inertial frame.

"Combined object": Energy momentum tensor $T_{\mu\nu}$

- Comes from the field-theoretic description of the matter field (given a Lagrangean, there is an "algorithm" to compute $T_{\mu\nu}$)

- If e_0 is timelike and unit: $T_{00} = T(e_0, e_0)$ is the energy density as measured by an observer moving in the direction of e_0

- Conservation of energy: $\nabla^\alpha T_{\alpha\beta} = 0$

So: The correct equation:
- 2^{nd} order in g
- geometric

$\Big\}$ It should involve $R_{\alpha\beta\gamma\delta}$, $g_{\alpha\beta}$

- Must have $T_{\mu\nu}$ on the right hand side

<u>Necessarily</u>:

$$\boxed{Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}} \qquad (G = c = 1)$$

- The only divergence-free contraction of the Riemann tensor
- Λ: cosmological constant, has units $1/[L]^2$

Originally: $\Lambda = 0$

(today: $\Lambda > 0$ at cosmological scales).

Why is 8π the correct constant on the right hand side?

So that we can recover Poisson's eqn in the <u>post-Newtonian limit</u>!

That is to say:
- Weak gravitational field: $g_{\mu\nu} = \eta_{\mu\nu} + \hat{g}_{\mu\nu}$, $\hat{g}_{\mu\nu} = O(\varepsilon_1)$
- Slow motion (observer & matter)

Free falling observer $x^\alpha(\tau)$ with

$$\dot{x}^0(\tau) = 1 + O(\varepsilon_2), \quad \dot{x}^i(\tau) = O(\varepsilon_2), \quad i = 1,2,3$$

And

$$T_{0i}, T_{ij} \ll T_{00} \quad (\text{so that } T_{00} = \rho + \text{l.o.t})$$

"lower order terms"

- Nearly static gravitational field:

$$(\partial_0 \hat{g}_{\alpha\beta} \approx 0)$$

Then: Geodesic equation:

$$\left(\frac{d}{d\tau} = \frac{d}{dx^0} + \text{l.o.t}\right) \qquad \ddot{x}^i(\tau) = -\Gamma^i_{00} + \dots = +\frac{1}{2}\partial_i \hat{g}_{00} + \dots$$

So in the "limit": The gravitational potential must satisfy (up to an additive constant):

$$\Phi = -\frac{1}{2} \hat{g}_{00}$$

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Moreover, from

① $\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$, if I take the trace:

$$R - \frac{4}{2} R = 8\pi \, \mathrm{tr}_g T$$

$$\overset{①}{\Longrightarrow} \mathrm{Ric}_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} \mathrm{tr}_g T \cdot g_{\mu\nu} \right)$$

$$\overset{\mu,\nu=0}{\Longrightarrow} \quad \mathrm{Ric}_{00} = 8\pi \left(T_{00} - \frac{1}{2} \mathrm{tr}_g T \cdot g_{00} \right) = 4\pi T_{00} + \dots$$
$$= 4\pi \rho + \dots$$

And $\mathrm{Ric}_{00} = \sum_{i=1}^{3} R_{0i0i}$

Since $R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\beta\delta} g_{\alpha\gamma} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\alpha\gamma} g_{\beta\delta} \right) + O((\partial g)^2)$

and $\partial_0 g_{\alpha\beta} \approx 0$

$$\mathrm{Ric}_{00} = -\frac{1}{2} \sum_{i=1}^{3} \partial^2_{ii} g_{00} + \dots = \Delta_{\mathbb{R}^3} \Phi + \dots$$

$\square$

We will consider the Einstein equations in any dimension.

Vacuum eqns: $\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$, (M^{n+1}, g)

If $n \geq 2$: $\Longleftrightarrow \mathrm{Ric}_{\mu\nu} = 0$ (In 1+1 dim: All Lorentzian manifolds satisfy this).

Unlike Newtonian gravity: There are non-trivial solutions in vacuum!

Simplest solution: Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$ ("The trivial" solution).

Causal structure:

$\forall p \in (\mathbb{R}^{n+1}, \eta)$, $J^+(p)$ is the same as $J_p^+ \subset T_p\mathbb{R}^{n+1}$ after identifying $T_p\mathbb{R}^{n+1} \approx \mathbb{R}^{n+1}$

$\partial J^+ = J^+ / I^+$. It is globally hyperbolic.

We would like to understand a bit better the causal structure of $(\mathbb{R}^{n+1}, \eta)$ asymptotically.

Note: If $g, \tilde{g}$ Lor. metrics on M with $g = \Omega \tilde{g}$, $\Omega > 0$, they have the <u>same</u> causal structure (e.g. $I^{\pm}(p)$, Cauchy hypersurfaces etc)

So we can try to <u>conformally compactify</u> spacetime, in order to "read off" the causal structure more easily.

1+1 Minkowski:

$$\eta = -dt^2 + dx^2$$

Double-null coordinates: $u = t - x$, $v = t + x$

$$\eta = -du\,dv$$

↖ Coordinate vector fields are null, so they capture better the geometry

Compactify: $(u, v) \longrightarrow (\tilde{u}, \tilde{v})$,

$$\tilde{u} = \tilde{u}(u) = \operatorname{Arctan} u$$

$$\tilde{v} = \tilde{v}(v) = \operatorname{Arctan} v$$

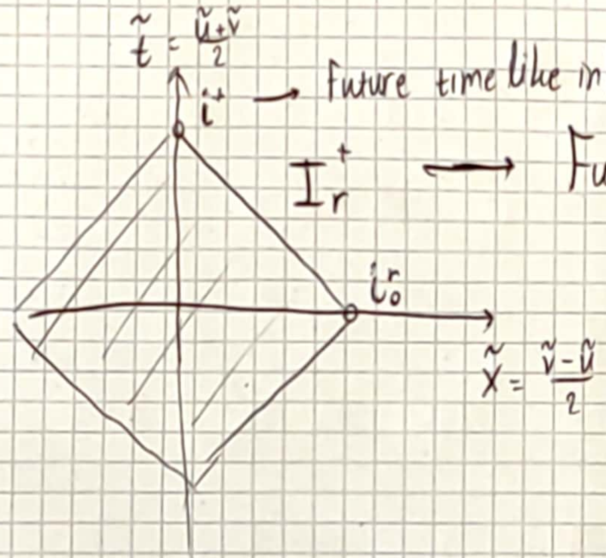
(Note: Any choice of function with finite range would do).

Then $\tilde{u}, \tilde{v} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$du = \frac{1}{\cos^2 \tilde{u}}\, d\tilde{u}, \qquad dv = \frac{1}{\cos^2 \tilde{v}}\, d\tilde{v}$$

$$\rightarrow \eta = -\frac{1}{\cos^2 \tilde{u} \cdot \cos^2 \tilde{v}}\, d\tilde{u}\, d\tilde{v} = \frac{1}{\cos^2 \tilde{u} \cdot \cos^2 \tilde{v}}\, \tilde{\eta},$$

Where $\tilde{\eta} = -d\tilde{u}\, d\tilde{v}$: Minkowski metric in the $(\tilde{u}, \tilde{v})$ plane.



$i^+ \rightarrow$ Future time like infinity

$I_r^+ \longrightarrow$ Future right null infinity

"endpoints" at infinity of original right moving fut. directed null geodesics.

If I choose different compactifying functions:
The image would still be a rectangle with null sides

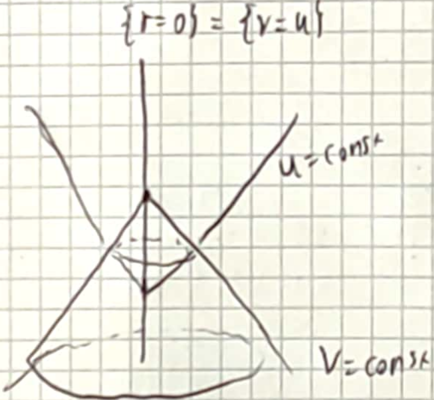
So the causal structure of the boundary at infinity is independent from those choices.

So what we did above: We constructed a map $F: (\mathbb{R}^{1+1}, \eta) \longrightarrow (N^{1+1}, \tilde{g}) = (\mathbb{R}^{1+1}, \tilde{\eta})$

conformal with pre-compact image

3+1 Minkowski

$$\eta = -dt^2 + dr^2 + r^2 g_{S^2}$$



$$\left.\begin{matrix} u = t-r \\ v = t+r \end{matrix}\right\}$$

$$r \geq 0 \Rightarrow \boxed{v \geq u}$$

$$\eta = -du\,dv + \frac{1}{4}(v-u)^2 g_{S^2}$$

As before: $\tilde{u} = \operatorname{Arctan} u$, $\tilde{v} = \operatorname{Arctan} v$

So: $(u, v, \underset{S^2}{\omega}) \longrightarrow (\tilde{u}, \tilde{v}, \omega)$

$$\eta = \frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}} \overbrace{\left(-d\tilde{u}\,d\tilde{v} + \frac{1}{4}\sin^2(\tilde{v}-\tilde{u})\, g_{S^2}\right)}^{\tilde{g}}$$

With $\tilde{v} \geq \tilde{u}$, $\tilde{v}, \tilde{u} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

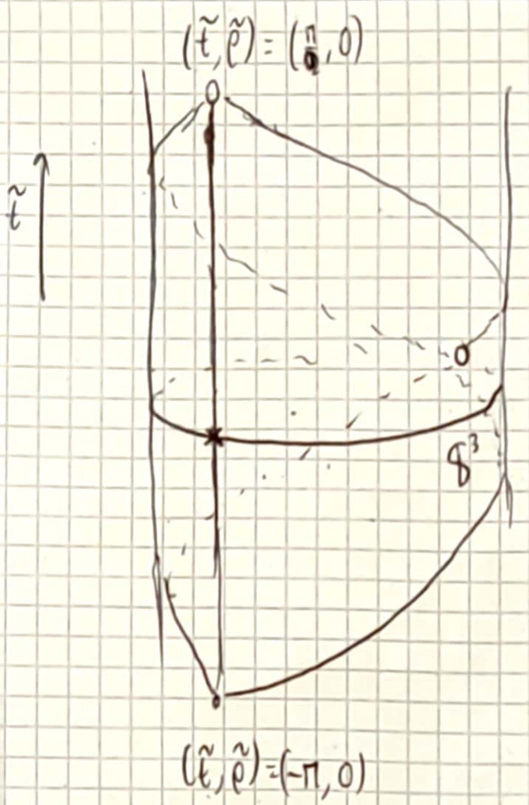
What metric is $\tilde{g}$?

If $\tilde{t} = \tilde{v} + \tilde{u}$, $\tilde{\rho} = \tilde{v} - \tilde{u}$. $\quad (0 \leq \tilde{\rho} < \pi)$

$$4\tilde{g} = -d\tilde{t}^2 + \underbrace{d\tilde{\rho}^2 + \sin^2\tilde{\rho}\; g_{S^2}}_{g_{S^3} \text{ in polar coordinates !!}}$$

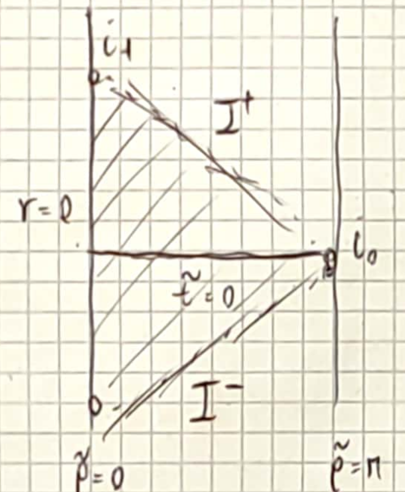
So the map we constructed maps $(\mathbb{R}^{3+1}, \eta)$ conformally into the domain of $(\mathbb{R} \times S^3, -d\tilde{t}^2 + g_{S^3})$ characterized by

$$\boxed{0 \leq \tilde{\rho} < \pi, \; |\tilde{t}| < \pi - \tilde{\rho}}$$



or, if I "forget" the S^2 direction

$(M / SO(3))$:



$\mathcal{I}^+$ is where "far away" radiation ends up.

Every point in this picture corresponds to a 2-sphere of $\mathbb{R}\times S^3$, <u>except</u> for $\tilde\rho = 0, \pi$

<u>Penrose diagram</u>:

Conformal image &

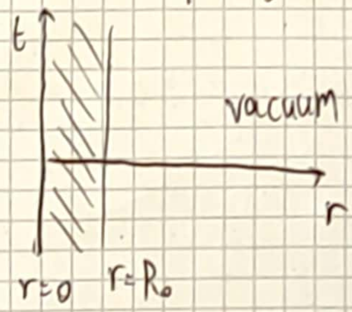
quotient by isometries.

- A point at $\mathcal{I}^+$: the point where "parallel" null straight lines end up in the future.

The Schwarzschild metric

Discovered in Jan. 1916 (a month after Einstein presented his eqns)

People were interested in the gravitational field of a static, sph. symmetric star:



Schwarzschild: For $M>0$, $r \geq R_0$:

$$g_M = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 g_{S^2} \quad \leftarrow (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Makes sense for any $M \in \mathbb{R}$ - we will only be interested in the case $M>0$.

Schwarzschil exterior spacetime: $(\mathcal{M}, g_M)$, $\mathcal{M} = \mathbb{R}_t \times (2M, +\infty)_r \times S^2_{\theta,\phi}$

M: "mass": Based on the effect on the motion of far away geodesics.

Note: • The vector field $T = \partial_t$ is Killing $(\partial_t g_{\mu\nu} = 0)$ and timelike (for the moment, we assume $r > 2M$)

$\rightsquigarrow$ The Schw. metric is static (T is hypersurface orthogonal)

- $SO(3)$ symmetry with S^2 orbits
- Asymptotically flat as $r \to +\infty$ (has $\mathcal{I}^{\pm}$ like Minkowski